HARMONIC ALMOST CONTACT STRUCTURES

E. Vergara-Diaz & C. M. Wood

Abstract. An almost contact metric structure is parametrized by a section σ of an associated homogeneous fibre bundle, and conditions for σ to be a harmonic section, and a harmonic map, are studied. These involve the characteristic vector field ξ, and the almost complex structure in the contact subbundle. Several examples are given where the harmonic section equations for σ reduce to those for ξ, regarded as a section of the unit tangent bundle. These include trans-Sasakian structures, and certain nearly cosymplectic structures. On the other hand, we obtain examples where ξ is harmonic but σ is not a harmonic section. Many of our examples arise by considering hypersurfaces of almost Hermitian manifolds, with the induced almost contact structure, and comparing the harmonic section equations for both structures.

1. Introduction

An almost contact metric structure on an orientable manifold M of odd dimension $2n+1$ may be regarded as a reduction of the structure group of M to $U(n)$ [15]. However the holonomy of such a manifold lies in $U(n)$ if and only if the almost contact structure is *cosymplectic* [1], which in particular precludes all contact metric structures. Thus, if the Riemannian metric g of M is fixed, reduction of holonomy to the natural structure group is of limited interest as a criterion for optimality in almost contact geometry. A more promising approach is to seek those almost contact structures whose characteristic vector field ξ is a harmonic section of the unit tangent bundle of M [28]. The study of such "harmonic unit vector fields" has been very active over the past few years, as evidenced by the extensive bibliography of [11]. In particular, contact metric manifolds whose characteristic vector field is harmonic were recently characterized in [22], and observed to constitute a very large class. This reflects the incomplete description of an almost contact structure by its characteristic vector field; equally significant is the almost complex structure J induced in the almost contact subbundle $\mathcal{D} = \xi^{\perp}$ by the

almost contact tensor ϕ. In order to weigh the entire almost contact structure it is natural to consider the corresponding section σ of the associated homogeneous fibre bundle $\pi\colon N \to M$ with fibre $SO(2n+1)/U(n)$, which may be regarded as the odd-dimensional analogue of the twistor bundle in almost Hermitian geometry. We then look for those σ which are *harmonic sections* of π. This methodology has already been applied to the study of almost complex structures [27, 8] and almost product structures [26, 6, 77] (of which harmonic unit vector fields are a special case), and recently a general theory of "harmonic reductions" was laid out in [29]. Following [29], in §2 we study the geometry of the *universal almost contact structure* for a Riemannian vector bundle, and in §3 we use this to obtain the harmonic section equations for σ, in terms of ξ and J. The calculations are more complicated and subtle than those for previously studied structures because the fibre of π is no longer an irreducible symmetric space; in fact it is neither irreducible nor symmetric. A naive expectation is that the harmonic section equations reduce to both ξ and J being harmonic. This turns out to be false: there is a first order coupling between ξ and J which in general prevents such a reduction (Theorem 3.2). We also derive the equations for σ to be a harmonic map (Theorem 3.4); as pointed out in [29] in general, these involve the curvature of (M, g).

In §4 we examine the simplest situation where the harmonic section equations for σ reduce to the harmonicity of ξ. This is when J is parallel, when we say that $\mathcal{D}$ is a *Kähler bundle*. Examples include all 3-dimensional almost contact structures, real hypersurfaces of Kähler manifolds (Theorem 4.3), and all trans-Sasakian structures (Theorem 4.8). The geometric data used for Theorem 4.8 yields an alternative proof of Marrero's structure theorem [19], using Bianchi's first identity (Theorem 4.7).

To study examples where $\mathcal{D}$ is not a Kähler bundle, in §5 we assume that M is an isometrically immersed hypersurface of an almost Hermitian manifold $\tilde{M}$, and equipped with the induced almost contact structure. We relate the harmonic section equations for the almost contact structure with those for the ambient almost Hermitian structure (Proposition 5.3). This allows us to study the canonical hypersurface embedding, where $\tilde{M} = M \times \mathbb{R}$ (Theorem 5.4). We use this to show that any *nearly cosymplectic* structure with parallel characteristic field is parametrized by a harmonic map. We also study the case when $\tilde{M}$ is nearly Kähler. Here we are able to characterize the harmonicity of the induced almost contact structure when M is a contact metric manifold (Theorem 5.9), or a totally umbilic submanifold (Theorem 5.10). In particular, σ is a harmonic map for the nearly cosymplectic S^5 (Theorem 5.11). We also exhibit examples where ξ is harmonic but σ is *not* a harmonic section; these include the *nearly Sasakian* S^5 [5].

Further examples of harmonic almost contact structures are considered in [24].

2. Universal Almost Contact Geomtry

For clarity we consider the general setup of an oriented vector bundle $\mathcal{E} \to M$ of odd rank $r = 2k + 1$, with connection ∇ and holonomy-invariant fibre metric $\langle\,,\rangle$. Thus we no longer require that M is odd-dimensional, although our primary application will be when this is the case and $\mathcal{E} = TM$.

Let $G = SO(2k+1)$ and $H = U(k)$, included as follows:

$$A + iB \mapsto \begin{pmatrix} A & -B & 0 \\ B & A & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

Let ϕ_0 be the skew-symmetric endomorphism with matrix:

$$\begin{pmatrix} \mathbb{O}_k & -\mathbb{I}_k & 0 \\ \mathbb{I}_k & \mathbb{O}_k & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Then ϕ_0 is an element of the Lie algebra $\mathfrak{g} = \mathrm{Skew}(\mathbb{R}^r)$ of G, and H and its Lie algebra $\mathfrak{h}$ may be characterized:

$$H = \{A \in G : A\,\phi_0\,A^{-1} = \phi_0\}, \qquad \mathfrak{h} = \{a \in \mathfrak{g} : [a, \phi_0] = 0\},$$

where $[\,,]$ is the commutator. Let $\mathfrak{m} \subset \mathfrak{g}$ be the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form of $\mathfrak{g}$, and let $\mathfrak{m}_1, \mathfrak{m}_2 \subset \mathfrak{m}$ be the following subspaces:

$$\mathfrak{m}_1 = \{a \in \mathfrak{g} : \{a, \phi_0\} = 0\}, \qquad \mathfrak{m}_2 = \{\{a, \eta_0 \otimes \xi_0\} : a \in \mathfrak{g}\},$$

where $\{\,,\}$ is the anticommutator, $\xi_0 = (0, \dots, 0, 1) \in \mathbb{R}^r$, and $\eta_0 \in (\mathbb{R}^r)^*$ is dual to ξ_0. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is an $\mathrm{Ad}(H)$-invariant splitting, with relations:

$$[\mathfrak{h}, \mathfrak{m}_1] \subset \mathfrak{m}_1, \qquad [\mathfrak{h}, \mathfrak{m}_2] \subset \mathfrak{m}_2$$

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}, \qquad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h} \oplus \mathfrak{m}_1, \qquad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2 \tag{2-1}$$

If $a \in \mathfrak{g}$ then the decomposition $a = a_{\mathfrak{h}} + a_{\mathfrak{m}_1} + a_{\mathfrak{m}_2}$ into components is:

$$a_{\mathfrak{h}} = -\tfrac{1}{2}\big(\phi_0\{a, \phi_0\} + a \circ (\eta_0 \otimes \xi_0)\big),$$

$$a_{\mathfrak{m}_1} = \tfrac{1}{2}\big(\phi_0[a, \phi_0] - a \circ (\eta_0 \otimes \xi_0)\big), \qquad a_{\mathfrak{m}_2} = \{a, \eta_0 \otimes \xi_0\} \tag{2-2}$$

The following H-equivariant isomorphisms will also be useful:

$$\mathfrak{h} \oplus \mathfrak{m}_1 \to \mathrm{Skew}(\mathbb{R}^{2k}) : a \mapsto \hat{a} = a\,|\,\mathbb{R}^{2k}$$

$$\mathfrak{m}_2 \to \mathbb{R}^{2k} : a_2 \mapsto \hat{a}_2 = a_2(\xi_0) \tag{2-3}$$

The first of these is an isometry, whereas the second is homothetic:

$$|a_2|^2 = 2|\hat{a}_2|^2 \tag{2-4}$$

Let $\mu\colon P \to M$ be the principal G-bundle of positively oriented orthonormal frames of $\mathcal{E}$. Then $\mu = \pi \circ \nu$ where $\nu\colon P \to P/H = N$ is a principal H-bundle, and $\pi\colon N \to M$ is naturally isomorphic to the homogeneous G/H-bundle associated to μ. Let $\boldsymbol{\mathcal{E}} = \pi^*\mathcal{E}$ be the pullback bundle, equipped with the pullback Riemannian structure. Thus $\boldsymbol{\mathcal{E}} \to N$ is a vector bundle with reduced structure group H. Since ξ_0 (resp. ϕ_0) is H-invariant there exists an induced unit section $\boldsymbol{\xi}$ of $\boldsymbol{\mathcal{E}}$ (resp. skew-symmetric endomorphism $\boldsymbol{\Phi}$ of $\boldsymbol{\mathcal{E}}$). Then $\ker \boldsymbol{\Phi} = \langle \boldsymbol{\xi} \rangle$, the line subbundle generated by $\boldsymbol{\xi}$, and we define $\boldsymbol{\mathcal{D}} = \operatorname{im} \boldsymbol{\Phi}$. The restriction $\boldsymbol{\Phi}|\boldsymbol{\mathcal{D}} = \boldsymbol{J}$, a complex structure in $\boldsymbol{\mathcal{D}}$.

Let $\mathfrak{H} \to N$ and $\mathfrak{M}_i \to N$ be the vector bundles associated to ν whose fibres are the H-modules $\mathfrak{h}$ and $\mathfrak{m}_i$. Each is a subbundle of $\operatorname{Skew}(\boldsymbol{\mathcal{E}})$:

$\mathfrak{H}$ (resp. $\mathfrak{M}_1$) commutes (resp. anticommutes) with $\boldsymbol{\Phi}$;

$\mathfrak{M}_2$ interchanges $\boldsymbol{\mathcal{D}}$ and $\langle \boldsymbol{\xi} \rangle$.

It follows that if α lies in $\mathfrak{H} \oplus \mathfrak{M}_1$ then $\alpha(\boldsymbol{\xi}) = 0$. The H-equivariant "hat" isomorphisms (2-3) induce isomorphisms of $\mathfrak{H}$ and $\mathfrak{M}_i$ with vector bundles $\widehat{\mathfrak{H}} \to N$ and $\widehat{\mathfrak{M}}_i \to N$ respectively, where $\widehat{\mathfrak{H}}$ (resp. $\widehat{\mathfrak{M}}_1$) are the subbundles of $\operatorname{Skew}(\boldsymbol{\mathcal{D}})$ which commute (resp. anticommute) with $\boldsymbol{J}$, and $\widehat{\mathfrak{M}}_2 = \boldsymbol{\mathcal{D}}$. If α lies in $\mathfrak{H} \oplus \mathfrak{M}_1$ then $\hat{\alpha} = \alpha|\boldsymbol{\mathcal{D}}$, whereas if α is in $\mathfrak{M}_2$ then $\hat{\alpha} = \alpha(\boldsymbol{\xi})$. For all sections α_i, β_i of $\mathfrak{M}_i$ it is easy to show that:

$$\alpha_2(u) = -\langle \hat{\alpha}_2, u \rangle \boldsymbol{\xi}, \quad \text{for all } u \in \boldsymbol{\mathcal{D}} \tag{2-5}$$

$$[\boldsymbol{\Phi}, \alpha_2]\hat{} = \boldsymbol{J}(\hat{\alpha}_2) \tag{2-6}$$

$$[\alpha_1, \alpha_2]\hat{} = \hat{\alpha}_1(\hat{\alpha}_2) \tag{2-7}$$

It follows from (2-6) that:

$$\text{if } \alpha_2 = [\boldsymbol{\Phi}, \beta_2] \ \text{ then } \ \beta_2 = -[\boldsymbol{\Phi}, \alpha_2]. \tag{2-8}$$

If $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \to N$ then the *homogeneous connection form* (cf. [29]) is the $\mathfrak{M}$-valued 1-form θ on N obtained by projecting the $\mathfrak{m}$-component of the connection form $\omega \in \Omega^1(P, \mathfrak{g})$. We note that $\nabla\boldsymbol{\Phi}$ is also an $\mathfrak{M}$-valued 1-form on N. For, if $\boldsymbol{\eta}$ is the section of $\boldsymbol{\mathcal{E}}^*$ dual to $\boldsymbol{\xi}$ then differentiating the identity

$$\boldsymbol{\Phi}^2 = -1 + \boldsymbol{\eta} \otimes \boldsymbol{\xi} \tag{2-9}$$

yields:

$$\{\nabla\boldsymbol{\Phi}, \boldsymbol{\Phi}\} = \nabla\boldsymbol{\eta} \otimes \boldsymbol{\xi} + \boldsymbol{\eta} \otimes \nabla\boldsymbol{\xi}$$

Hence:

$$\mathbf{\Phi}\{\nabla\mathbf{\Phi}, \mathbf{\Phi}\} + \nabla\mathbf{\Phi} \circ (\boldsymbol{\eta} \otimes \boldsymbol{\xi}) = \boldsymbol{\eta} \otimes (\mathbf{\Phi}\,\nabla\boldsymbol{\xi} + \nabla\mathbf{\Phi}(\boldsymbol{\xi})) = \boldsymbol{\eta} \otimes \nabla(\mathbf{\Phi}\boldsymbol{\xi}) = 0,$$

and it follows from (2-2) that the $\mathfrak{H}$-component of $\nabla\mathbf{\Phi}$ vanishes. Both θ and $\nabla\mathbf{\Phi}$ therefore split into $\mathfrak{M}_i$-components:

$$\theta = \theta_1 + \theta_2, \qquad \nabla\mathbf{\Phi} = (\nabla\mathbf{\Phi})_1 + (\nabla\mathbf{\Phi})_2$$

2.1 Lemma.

(a) $\theta_1 = \frac{1}{2}\mathbf{\Phi} \circ (\nabla\mathbf{\Phi})_1;$ (b) $\theta_2 = [\mathbf{\Phi}, (\nabla\mathbf{\Phi})_2]$

Proof. Since $\mathbf{\Phi}$ lifts to the (constant) H-equivariant map $\phi_0 \colon P \to \mathfrak{h}$ we have (cf. [29, Proposition 4.1]):

$$\nabla\mathbf{\Phi} = [\theta, \mathbf{\Phi}] = [\theta_1, \mathbf{\Phi}] + [\theta_2, \mathbf{\Phi}] = -2\mathbf{\Phi} \circ \theta_1 + [\theta_2, \mathbf{\Phi}],$$

and therefore:

$$(\nabla\mathbf{\Phi})_1 = -2\mathbf{\Phi} \circ \theta_1, \qquad (\nabla\mathbf{\Phi})_2 = [\theta_2, \mathbf{\Phi}],$$

from which (a) follows directly, and (b) follows via (2-8). □

An expression for θ in terms of $\boldsymbol{J}$ and $\boldsymbol{\xi}$ is obtained by applying the "hat" isomorphisms to Lemma 2.1. Let $\bar{\nabla}$ denote the connection in $\mathcal{D} \to N$ obtained by orthogonally projecting ∇, and note that $\nabla\boldsymbol{\xi}$ is $\mathcal{D}$-valued since $\boldsymbol{\xi}$ is a unit section.

2.2 Proposition.

(a) $\hat{\theta}_1 = \frac{1}{2}\boldsymbol{J}\,\bar{\nabla}\boldsymbol{J};$ (b) $\hat{\theta}_2 = \nabla\boldsymbol{\xi}$

Proof.

(a) If $A \in TN$ and u is a section of $\mathcal{D}$ then since $\mathfrak{M}_2$ swaps $\mathcal{D}$ and $\langle\boldsymbol{\xi}\rangle$ it follows from Lemma 2.1 (a) that:

$$\begin{aligned}
2\hat{\theta}_1(A)(u) = 2\theta_1(A)(u) &= \mathbf{\Phi}(\nabla_A\mathbf{\Phi})_1(u) = \mathbf{\Phi}\,\nabla_A\mathbf{\Phi}(u) \\
&= \mathbf{\Phi}\,\nabla_A(\boldsymbol{J}u) + \nabla_A u - \langle\nabla_A u, \boldsymbol{\xi}\rangle\boldsymbol{\xi}, \quad \text{by (2-9)} \\
&= \boldsymbol{J}\,\bar{\nabla}_A(\boldsymbol{J}u) + \bar{\nabla}_A u = \boldsymbol{J}(\bar{\nabla}_A(\boldsymbol{J}u) - \boldsymbol{J}\,\bar{\nabla}_A u) = \boldsymbol{J}\,\bar{\nabla}_A\boldsymbol{J}(u).
\end{aligned}$$

(b) By Lemma 2.1 (b) and (2-6):

$$\begin{aligned}
\hat{\theta}_2 &= \boldsymbol{J}(\nabla\mathbf{\Phi})_2\hat{} = \mathbf{\Phi}(\nabla\mathbf{\Phi})_2(\boldsymbol{\xi}) = \mathbf{\Phi}\,\{\nabla\mathbf{\Phi}, \boldsymbol{\eta} \otimes \boldsymbol{\xi}\}\,(\boldsymbol{\xi}), \quad \text{by (2-2)} \\
&= \mathbf{\Phi}\,(\nabla\mathbf{\Phi}(\boldsymbol{\xi}) + \langle\nabla\mathbf{\Phi}(\boldsymbol{\xi}), \boldsymbol{\xi}\rangle\boldsymbol{\xi}) = -\mathbf{\Phi}^2\,\nabla\boldsymbol{\xi} = \nabla\boldsymbol{\xi}, \quad \text{by (2-9).} \qquad \square
\end{aligned}$$

The $\mathfrak{h}$-component of ω is a connection form in $\nu \colon P \to N$, and induces linear connections in the associated vector bundles $\mathfrak{M}, \mathfrak{M}_i \to N$, each of which we denote by ∇^c and call the *canonical connection* for the bundle in question. For all sections α_i of $\mathfrak{M}_i$:

$$\nabla^c(\alpha_1 + \alpha_2) = \nabla^c\alpha_1 + \nabla^c\alpha_2 \tag{2-10}$$

2.3 Proposition. *For all sections α_i of $\mathfrak{M}_i$ we have:*

(a) $(\nabla^c \alpha_1)\hat{} = \frac{1}{2} \boldsymbol{J} \, [\bar{\nabla} \hat{\alpha}_1, \boldsymbol{J}];$ (b) $(\nabla^c \alpha_2)\hat{} = \bar{\nabla} \hat{\alpha}_2 - \frac{1}{2} \boldsymbol{J} \, \bar{\nabla} \boldsymbol{J}(\hat{\alpha}_2)$

Proof. If α is a section of $\mathfrak{M}$ then by the structure equations (cf. [29, Proposition 2.7]):

$$\nabla \alpha \,=\, \nabla^c \alpha + [\theta, \alpha] \,=\, \nabla^c \alpha + [\theta_1, \alpha] + [\theta_2, \alpha] \tag{2-11}$$

(a) By (2-11) and (2-1), $\nabla^c \alpha_1$ is the $\mathfrak{M}_1$-component of $\nabla \alpha_1$, and hence by (2-2):

$$2 \, \nabla^c \alpha_1 \,=\, \boldsymbol{\Phi}[\nabla \alpha_1, \boldsymbol{\Phi}] - \nabla \alpha_1 \circ (\boldsymbol{\eta} \otimes \boldsymbol{\xi})$$

Therefore if $u \in \mathcal{D}$ then:

$$2(\nabla^c \alpha_1)\hat{}(u) \,=\, 2 \, \nabla^c \alpha_1(u) \,=\, \boldsymbol{\Phi}[\nabla \alpha_1, \boldsymbol{\Phi}](u) \,=\, \boldsymbol{\Phi} \, \nabla \alpha_1(\boldsymbol{\Phi} u) - \boldsymbol{\Phi}^2 \, \nabla \alpha_1(u)$$
$$=\, \boldsymbol{\Phi}\big(\bar{\nabla} \hat{\alpha}_1(\boldsymbol{J} u) + \langle \nabla \alpha_1(\boldsymbol{J} u), \boldsymbol{\xi} \rangle \boldsymbol{\xi}\big) + \nabla \alpha_1(u) - \langle \nabla \alpha_1(u), \boldsymbol{\xi} \rangle \boldsymbol{\xi}$$
$$=\, \boldsymbol{J} \, \bar{\nabla} \hat{\alpha}_1(\boldsymbol{J} u) - \bar{\nabla} \hat{\alpha}_1(u) \,=\, \boldsymbol{J} \, [\bar{\nabla} \hat{\alpha}_1, \boldsymbol{J}](u)$$

(b) It follows from (2-11), (2-1) and (2-2) that:

$$\nabla^c \alpha_2 + [\theta_1, \alpha_2] \,=\, \{\nabla \alpha_2, \boldsymbol{\eta} \otimes \boldsymbol{\xi}\}$$

Each term in this equation is $\mathfrak{M}_2$-valued, so for all A in TN we have:

$$(\nabla^c_A \alpha_2)\hat{} \,=\, \{\nabla_A \alpha_2, \boldsymbol{\eta} \otimes \boldsymbol{\xi}\}(\boldsymbol{\xi}) - \hat{\theta}_1(A)(\hat{\alpha}_2), \quad \text{by (2-7)}$$
$$=\, \nabla_A \alpha_2(\boldsymbol{\xi}) - \tfrac{1}{2} \, \boldsymbol{J} \, \bar{\nabla}_A \boldsymbol{J}(\hat{\alpha}_2), \quad \text{by Proposition 2.2 (a)},$$

since $\nabla_A \alpha_2$ is skew-symmetric. Now note that:

$$\nabla \alpha_2(\boldsymbol{\xi}) \,=\, \nabla \hat{\alpha}_2 - \alpha_2(\nabla \boldsymbol{\xi}) \,=\, \nabla \hat{\alpha}_2 + \langle \hat{\alpha}_2, \nabla \boldsymbol{\xi} \rangle \boldsymbol{\xi}, \quad \text{by (2-5)}$$
$$=\, \nabla \hat{\alpha}_2 - \langle \nabla \hat{\alpha}_2, \boldsymbol{\xi} \rangle \boldsymbol{\xi} \,=\, \bar{\nabla} \hat{\alpha}_2 \qquad\qquad \square$$

Finally, we recall from [29] that the *homogeneous curvature form* is the $\mathfrak{M}$-valued 2-form Θ on N obtained by projecting the $\mathfrak{m}$-component $\Omega_{\mathfrak{m}}$ of the curvature form $\Omega \in \Omega^2(P, \mathfrak{g})$. Then $\Theta = \Theta_1 + \Theta_2$. Let R be the curvature tensor of ∇. Since Θ_i (resp. R) is the projection to N (resp. M) of $\Omega_{\mathfrak{m}_i}$ (resp. 2Ω) it follows from (2-2) that:

$$4\Theta_1 \,=\, \boldsymbol{\Phi}[\pi^* R, \boldsymbol{\Phi}] - \pi^* R \circ (\boldsymbol{\eta} \otimes \boldsymbol{\xi}), \qquad 2\Theta_2 \,=\, \{\pi^* R, \boldsymbol{\eta} \otimes \boldsymbol{\xi}\} \tag{2-12}$$

3. The Harmonic Section Equations

Let $\sigma \colon M \to N$ be a section of π. The corresponding almost contact structure in $\mathcal{E}$ is obtained by pulling back the universal almost contact structure:

$$\phi = \sigma^*\mathbf{\Phi}, \qquad \xi = \sigma^*\boldsymbol{\xi}$$

Then $\mathcal{D} = \sigma^*\boldsymbol{\mathcal{D}}$ and $J = \sigma^*\boldsymbol{J}$. Let $Q \to M$ be the principal H-subbundle of $P \to M$ comprising all ϕ-*adapted* positively oriented orthonormal frames of $\mathcal{E}$, and let $\mathfrak{M} \to M$ (resp. $\mathfrak{M}_i \to M$) be the associated vector bundle with fibre $\mathfrak{m}$ (resp. $\mathfrak{m}_i$). Then:

$$\mathfrak{M} = \sigma^*\boldsymbol{\mathfrak{M}}, \qquad \mathfrak{M}_i = \sigma^*\boldsymbol{\mathfrak{M}}_i$$

Thus $\mathfrak{M}_1$ (resp. $\mathfrak{M}_2$) is the subbundle of $\mathrm{Skew}(\mathcal{E})$ which anticommutes with ϕ (resp. swaps $\mathcal{D}$ and $\langle\xi\rangle$). Applying the "hat" isomorphisms yields the bundle $\widehat{\mathfrak{M}}_1$ of skew-symmetric endomorphisms of $\mathcal{D}$ which anticommute with J, and $\widehat{\mathfrak{M}}_2 = \mathcal{D}$. Define $\psi = \sigma^*\theta$, an $\mathfrak{M}$-valued 1-form on M. Pulling back Proposition 2.2 by σ yields:

$$\hat{\psi}_1 = \tfrac{1}{2}J\,\bar{\nabla}J, \qquad \hat{\psi}_2 = \nabla\xi \tag{3-1}$$

In particular, σ is horizontal ($\psi = 0$) if and only if ξ and J are parallel.

Pulling back ∇^c by σ yields connections in $\mathfrak{M}, \mathfrak{M}_i \to M$, which will also be denoted by ∇^c; they are simply the linear connections in these bundles associated to the H-connection form $\omega_{\mathfrak{h}}|_{TQ}$. For all sections α_i of $\mathfrak{M}_i$ it follows from Proposition 2.3 that:

$$(\nabla^c\alpha_1)\hat{} = \tfrac{1}{2}J\,[\bar{\nabla}\hat{\alpha}_1, J], \qquad (\nabla^c\alpha_2)\hat{} = \bar{\nabla}\hat{\alpha}_2 - \tfrac{1}{2}J\,\bar{\nabla}J(\hat{\alpha}_2) \tag{3-2}$$

We note that $\nabla^c\psi = \nabla^c\psi_1 + \nabla^c\psi_2$ by (2-10), and introduce 2-covariant tensors on M:

$$\langle\nabla\xi \otimes \nabla\xi\rangle(X,Y) = \langle\nabla_X\xi, \nabla_Y\xi\rangle, \qquad (\bar{\nabla}J \otimes \nabla\xi)(X,Y) = \bar{\nabla}_X J(\nabla_Y\xi),$$

the first (resp. second) of which is symmetric (resp. $\mathcal{D}$-valued).

3.1 Proposition.
(a) $(\nabla^c\psi_1)\hat{} = \tfrac{1}{4}[J, \bar{\nabla}^2 J]$
(b) $(\nabla^c\psi_2)\hat{} = \nabla^2\xi + \langle\nabla\xi \otimes \nabla\xi\rangle\xi - \tfrac{1}{2}J\,\bar{\nabla}J \otimes \nabla\xi$

Proof.
(a) From the first of equations (3-2):

$$4(\nabla^c\psi_1)\hat{} = 2\,J\big[\bar{\nabla}\hat{\psi}_1, J\big] = J\big[\bar{\nabla}(J\,\bar{\nabla}J), J\big], \quad \text{by (3-1)}$$

$$= J\big[\bar{\nabla}J \circ \bar{\nabla}J, J\big] + J\big[J\,\bar{\nabla}^2 J, J\big] = J\big[J\,\bar{\nabla}^2 J, J\big], \quad \text{since } \{\bar{\nabla}J, J\} = 0,$$

$$= J^2 \circ \bar{\nabla}^2 J \circ J - J^3 \circ \bar{\nabla}^2 J = J \circ \bar{\nabla}^2 J - \bar{\nabla}^2 J \circ J = [J, \bar{\nabla}^2 J].$$

(b)   From the second of equations (3-2), and (3-1):

$$(\nabla^c \psi_2)\hat{} = \bar{\nabla}\hat{\psi}_2 - \tfrac{1}{2}J\bar{\nabla}J(\hat{\psi}_2) = \bar{\nabla}(\nabla\xi) - \tfrac{1}{2}J\bar{\nabla}J \otimes \nabla\xi,$$

and since ξ is a unit vector:

$$\bar{\nabla}_X(\nabla\xi)(Y) = \nabla_X(\nabla\xi)(Y) - \langle \nabla_X(\nabla\xi)(Y), \xi\rangle\xi$$
$$= \nabla^2_{X,Y}\xi - \langle \nabla_X\nabla_Y\xi, \xi\rangle\xi = \nabla^2_{X,Y}\xi + \langle \nabla_X\xi, \nabla_Y\xi\rangle\xi. \qquad \square$$

Let $\mathcal{V} \to N$ be the π-vertical subbundle of TN. The *vertical tension field* of σ is:

$$\tau^v\sigma = \operatorname{Tr}\nabla^v d^v\sigma = \nabla^v_{E_i}d^v\sigma(E_i),$$

where $\{E_i\}$ is any orthonormal frame of TM (here and henceforward the summation convection is in force), and $d^v\sigma$ (resp. ∇^v) is the $\mathcal{V}$-component of $d\sigma$ (resp. the Levi-Civita connection of N). (The Riemannian metric on N is obtained by horizontally lifting g, and supplementing with the fibre metric in $\mathcal{V}$ induced by the G-invariant metric on G/H.) Then σ is a *harmonic section* of π if and only if $\tau^v\sigma = 0$. From [29], the restriction $\theta|_{\mathcal{V}}$ is a canonical vector bundle isomorphism $I: \mathcal{V} \to \mathfrak{M}$, and $I \circ d^v\sigma = \psi$. Because G/H is not a symmetric space, I is not connection-preserving; however since G/H is naturally reductive we have:

$$I \circ \tau^v\sigma = -\delta^c\psi = \nabla^c_{E_i}\psi(E_i),$$

by [29, Theorem 3.2]. We have $I = I_1 + I_2$ from the splitting of $\mathfrak{M}$, and by (2-10):

$$I_i \circ \tau^v\sigma = -\delta^c\psi_i$$

We also recall the definitions of the *rough Laplacians*:

$$\bar{\nabla}^*\bar{\nabla}J = -\bar{\nabla}^2_{E_i,E_i}J, \qquad \nabla^*\nabla\xi = -\nabla^2_{E_i,E_i}\xi$$

In general, a Hermitian structure J in a Riemannian vector bundle is said to be *harmonic* if the corresponding section of the associated twistor bundle is a harmonic section, and by [27] this is the case precisely when J commutes with its rough Laplacian. We also say that ξ *is harmonic* if ξ is a harmonic section of the unit sphere bundle $U\mathcal{E} \to M$, and by [28] this is the case precisely when:

$$\nabla^*\nabla\xi = |\nabla\xi|^2\xi \qquad (3\text{-}3)$$

We define a section $T(\phi)$ of $\mathcal{D}$ as follows:

$$T(\phi) = \operatorname{Tr}(\bar{\nabla}J \otimes \nabla\xi) = \bar{\nabla}_{E_i}J(\nabla_{E_i}\xi)$$

and abbreviate:

$$\bar{\tau}(J) = \tfrac{1}{4}[\bar{\nabla}^*\bar{\nabla}J, J]$$

Thus $\bar{\tau}(J)$ is a skew-symmetric endomorphism of $\mathcal{D}$ which anticommutes with J. The following result is an immediate consequence of Proposition 3.1.

3.2 Theorem.

(a) $(I_1 \tau^v \sigma)\hat{} = \bar{\tau}(J)$

(b) $(I_2 \tau^v \sigma)\hat{} = -\nabla^*\nabla\xi + |\nabla\xi|^2\xi - \frac{1}{2}JT(\phi)$

Therefore σ is a harmonic section if and only if J is harmonic and:

$$\nabla^*\nabla\xi = |\nabla\xi|^2\xi - \frac{1}{2}JT(\phi)$$

If ξ and J are harmonic then σ is a harmonic section precisely when $T(\phi) = 0$.

We now define an $\mathfrak{M}$-valued 2-form $\Psi = \sigma^*\Theta$ on M. Let $R_{\mathcal{D}}$ denote the curvature-type tensor in $\mathcal{D}$ obtained by orthogonally projecting R:

$$R_{\mathcal{D}}(X,Y)u = R(X,Y)u - \langle R(X,Y)u, \xi\rangle\xi, \quad \text{for all } u \in \mathcal{D}.$$

If $\bar{R}$ is the curvature tensor of the connection $\bar{\nabla}$ in $\mathcal{D}$ then:

$$\bar{R} = R_{\mathcal{D}} + r(\nabla\xi, \nabla\xi), \tag{3-4}$$

where r is the following curvature-type operator in $\mathcal{E}$:

$$r(u,v)w = \langle v,w\rangle u - \langle u,w\rangle v, \quad \text{for all } u,v,w \in \mathcal{E}. \tag{3-5}$$

3.3 Proposition.

(a) $\hat{\Psi}_1 = \frac{1}{4}J[R_{\mathcal{D}}, J];$ \qquad (b) $\hat{\Psi}_2(X,Y) = \frac{1}{2}R(X,Y)\xi$

Proof. (a) Pulling back equation the first of equations (2-12) by σ yields:

$$4\hat{\Psi}_1(X,Y)u = 4\Psi_1(X,Y)u = \phi[R(X,Y),\phi]u = J[R_{\mathcal{D}}(X,Y),J]u$$

(b) From the second of equations (2-12):

$$2\hat{\Psi}_2(X,Y) = 2\Psi_2(X,Y)\xi = \{R(X,Y),\eta\otimes\xi\}\xi = R(X,Y)\xi. \qquad \square$$

We recall from [29] that σ is a *harmonic map* precisely when σ is a harmonic section and the following 1-form $\langle\psi,\Psi\rangle$ on M vanishes:

$$\langle\psi,\Psi\rangle(X) = \langle\psi\otimes\Psi\rangle(E_i, E_i, X) = \langle\psi(E_i),\Psi(E_i,X)\rangle$$
$$= \langle\psi_1,\Psi_1\rangle(X) + \langle\psi_2,\Psi_2\rangle(X)$$

If $\bar{\sigma}$ is the section of the twistor bundle of $\mathcal{D}$ parametrizing J then from [29]:

$$\langle\bar{\psi},\bar{\Psi}\rangle(X) = \langle\bar{R}(E_i,X), J\bar{\nabla}_{E_i}J\rangle, \tag{3-6}$$

and if ξ is harmonic then $\xi\colon M \to U\mathcal{E}$ is a harmonic map precisely when [10]:

$$\langle R(E_i,X)\xi, \nabla_{E_i}\xi\rangle = 0 \tag{3-7}$$

Use of Proposition 3.3 with (3-2), (3-4) and (2-4) yields the following:

3.4 Theorem. *For all X in TM we have:*

(a) $\quad \langle \psi_1, \Psi_1 \rangle(X) = \frac{1}{4} \langle R(E_i, X), J\bar{\nabla}_{E_i} J \rangle = \langle \bar{\psi}, \bar{\Psi} \rangle(X) + \frac{1}{2} \langle JT(\phi), \nabla_X \xi \rangle$

$\qquad\qquad\qquad = \langle \bar{\psi}, \bar{\Psi} \rangle(X) - \langle \nabla^* \nabla \xi, \nabla_X \xi \rangle, \quad$ *if σ is a harmonic section.*

(b) $\quad \langle \psi_2, \Psi_2 \rangle(X) = \langle R(E_i, X)\xi, \nabla_{E_i} \xi \rangle$

If σ is a harmonic section, and $\bar{\sigma}$ and ξ are harmonic maps, then σ is a harmonic map.

4. Kähler Contact Bundle

There are two partial reductions of the holonomy of $\mathcal{E}$, which yield significant simplifications to Theorems 3.2 and 3.4.

Definition. A Riemannian vector bundle equipped with a parallel Hermitian structure is said to be a *Kähler bundle*.

4.1 Theorem.
(a) *If $(\mathcal{D}, \bar{\nabla}, J)$ is a Kähler bundle then σ is a harmonic section (resp. harmonic map) precisely when ξ is a harmonic section of $U\mathcal{E}$ (resp. harmonic map).*
(b) *If ξ is parallel then σ is a harmonic section (resp. harmonic map) precisely when $\bar{\sigma}$ is a harmonic section (resp. harmonic map).*

Henceforward we assume that M has odd dimension $2n + 1$, and $\mathcal{E} = TM$, with the Levi Civita connection. There are two "obvious" cases when $\mathcal{D}$ is a Kähler bundle. If M is an oriented (real) hypersurface of an almost Hermitian manifold $(\tilde{M}, g, J)$, with compatible unit normal ν, then there is an induced almost contact structure:

$$\xi = -J\nu, \qquad \phi X = JX - \eta(X)\nu \qquad\qquad (4\text{-}1)$$

The induced almost complex structure in $\mathcal{D}$ is thus the restriction of J.

4.2 Lemma. *The induced almost complex structure in $\mathcal{D}$ satisfies:*

$$\bar{\nabla} J = \tilde{\nabla} J - \langle \tilde{\nabla} J, \xi \rangle \xi - \langle \tilde{\nabla} J, \nu \rangle \nu$$

Proof. This follows from the simple identity, for all sections Z of $\mathcal{D}$:

$$\bar{\nabla} Z = \tilde{\nabla} Z - \langle \tilde{\nabla} Z, \xi \rangle \xi - \langle \tilde{\nabla} Z, \nu \rangle \nu \qquad\qquad \square$$

4.3 Theorem. *Suppose either of the following holds:*
(a) *M is a 3-dimensional almost contact manifold.*
(b) *M is an oriented real hypersurface of a Kähler manifold, equipped with the induced almost contact structure.*
Then $\mathcal{D}$ is a Kähler bundle. Hence σ is a harmonic section (resp. harmonic map) if and only if ξ is a harmonic unit vector field (resp. harmonic map).

Proof. Part (b) follows directly from Lemma 4.2. For (a) note that if $Z \in \mathcal{D}_x$ is a unit vector, then (Z, JZ) is an orthonormal basis of $\mathcal{D}_x$, and for all $X \in \mathcal{D}_x$ we have:

$$\langle \bar{\nabla}_X J(Z), Z \rangle = 0, \quad 2\langle \bar{\nabla}_X J(Z), JZ \rangle = \langle \bar{\nabla}_X J(Z), JZ \rangle - \langle JZ, \bar{\nabla}_X J(Z) \rangle = 0. \qquad \square$$

4.4 Example. Harmonic unit vector fields on 3-manifolds have been extensively studied, notably in [12]; in particular, each Thurston geometry has a natural harmonic unit field. Those on $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, *Nil*, *Sol* and $\widetilde{SL_2}$ are invariant under the full isometry group. Every discrete subgroup of Euclidean isometries acting freely on $\mathbb{R}^3$ leaves a direction invariant, and the Hopf field on S^3 is invariant under all finite subgroups of isometries which act freely [23]. Hence by Theorem 4.3 (a), any 3-manifold with non-hyperbolic geometric structure has a natural harmonic almost contact structure.

4.5 Example. Let $M = S^{2n+1}$, included in $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ as the unit sphere, and oriented by the unit outward-pointing normal. Then $-\xi$ is the standard Hopf vector field, and the induced almost contact structure is the standard Sasakian structure on S^{2n+1}. It is well-known that ξ is a harmonic unit vector field [25], and indeed a harmonic map [16]; therefore σ is a harmonic map by Theorem 4.3 (b).

Our next example generalizes Example 4.5 in a different way. It follows from

$$\bar{\nabla}_X J(Z) = \nabla_X \phi(Z) - \langle \nabla_X \phi(Z), \xi \rangle \xi, \qquad Z \in \mathcal{D}, \tag{4-2}$$

that $\mathcal{D}$ is a Kähler bundle if and only if $\nabla_X \phi \in \mathfrak{M}_2$ for all X in TM. This is clearly the case for a *trans-Sasakian structure of type* (α, β) [21, 4]:

$$\nabla_X \phi(Y) = \alpha r(\xi, X)Y + \beta r(\xi, \phi X)Y,$$

where $\alpha, \beta \colon M \to \mathbb{R}$ are smooth functions, and r is defined in (3-5). If $\alpha = 0$ (resp. $\beta = 0$) the structure is said to be β-*Kenmotsu* (resp. α-*Sasakian*). The characteristic vector field of a trans-Sasakian manifold satisfies [9, 17]:

$$\nabla_X \xi = -\alpha \phi X - \beta \phi^2 X \tag{4-3}$$

$$\nabla^2_{X,Y} \xi = (\alpha^2 - \beta^2)\eta(Y)X + 2\alpha\beta\eta(Y)\phi X - (X.\alpha)\phi Y - (X.\beta)\phi^2 Y \tag{4-4}$$

$$R(X,Y)\xi = \phi r(X,Y)\nabla\alpha + \phi^2 r(X,Y)\nabla\beta + (\alpha^2 - \beta^2)r(X,Y)\xi + 2\alpha\beta\phi r(X,Y)\xi \tag{4-5}$$

$$\mathrm{Ric}(\xi) = \phi \nabla\alpha - \phi^2 \nabla\beta - 2n \nabla\beta + 2n(\alpha^2 - \beta^2)\xi, \tag{4-6}$$

where $\nabla\alpha, \nabla\beta$ are the gradient vectors. Of particular interest is the 2-form $\mathcal{R}(\Phi)$, where $\mathcal{R}\colon \Omega^2(M) \to \Omega^2(M)$ is the *curvature operator* and Φ is the *fundamental 2-form:*

$$\Phi(X, Y) \,=\, \langle X, \phi Y \rangle$$

If $\{F_i\}$ is an orthonormal frame of $\mathcal{D}$, extended to a frame $\{E_i\}$ of TM with $E_{2n+1} = \xi$, then:

$$\mathcal{R}(\Phi)(X, Y) \,=\, -\tfrac{1}{2}\langle R(X, Y)E_i, \phi E_i \rangle \,=\, -\tfrac{1}{2}\langle R(X, Y)F_i, \phi F_i \rangle \tag{4-7}$$

$$=\, \langle R(X, F_i)\phi F_i, Y \rangle, \tag{4-8}$$

where (4-8) follows from (4-7) by the symmetries of R, including Bianchi's first identity. Let $\mathcal{R}(\Phi)^\sharp$ denote the corresponding endomorphism field. Using (4-5) in each of (4-7) and (4-8) yields two expressions for $\mathcal{R}(\Phi)$, and an alternative proof of [19, Theorem 4.1].

4.6 Lemma. *On a trans-Sasakian manifold we have:*
(a) $\mathcal{R}(\Phi)^\sharp(\xi) = \phi^2 \nabla\alpha - \phi \nabla\beta$
(b) $\mathcal{R}(\Phi)^\sharp(\xi) = -\phi^2 \nabla\alpha - \phi \nabla\beta - 2n \nabla\alpha - 4n\alpha\beta\xi$

4.7 Theorem. *For all trans-Sasakian structures:* $d\alpha(\xi) = -2\alpha\beta.$
If $\dim M \geqslant 5$ *then* $\alpha\beta = 0$ *and* $\nabla\alpha = 0$. *Thus if* M *is connected it is either* α-*Sasakian with* α *constant, or* β-*Kenmotsu.*

Proof. Since $\mathcal{R}(\Phi)$ is skew-symmetric it follows from Lemma 4.6 (b) that:

$$0 \,=\, \mathcal{R}(\Phi)(\xi, \xi) \,=\, -2n\langle \nabla\alpha, \xi \rangle \,-\, 4n\alpha\beta,$$

and therefore:

$$\langle \nabla\alpha, \xi \rangle \,=\, -2\alpha\beta \tag{4-9}$$

Now, equating the two identities of Lemma 4.6 yields:

$$0 \,=\, \phi^2 \nabla\alpha + n \nabla\alpha + 2n\alpha\beta\xi \,=\, \phi^2 \nabla\alpha - n\phi^2 \nabla\alpha + n\langle \nabla\alpha, \xi \rangle \xi + 2n\alpha\beta\xi$$

$$=\, (1 - n)\phi^2 \nabla\alpha + 2(n - 1)\alpha\beta\xi, \quad \text{using (4-9)}.$$

Therefore if $n > 1$ then the $\mathcal{D}$-component of $\nabla\alpha$ vanishes and $\alpha\beta = 0$. It then follows from (4-9) that $\nabla\alpha = 0$. $\square$

Remark. Identity (4-9) was noted in [9].

Definition. A vector field (resp. 1-form) on M is *characteristic* if it is proportional to ξ (resp. η).

4.8 Theorem. *Suppose M is trans-Sasakian of type (α, β). Then σ is a harmonic section if and only if:*

$$\nabla\alpha = -\phi\,\nabla\beta - 2\alpha\beta\xi,$$

and σ is a harmonic map if and only if in addition:

$$\xi.(\beta^2) = 2\beta(\alpha^2 - \beta^2)$$

(a)   *If M is α-Sasakian then σ is a harmonic section if and only if α is locally constant, in which case σ is a harmonic map.*

(b)   *If M is β-Kenmotsu, or $\dim M \geqslant 5$, then σ is a harmonic section if and only if $\nabla\beta$ is characteristic, and σ is a harmonic map if and only if $\nabla\beta = -\beta^2\xi$. If M is compact β-Kenmotsu then σ is a harmonic map precisely when M is cosymplectic.*

Proof. From (4-4):

$$\nabla^*\nabla\xi = \phi\,\nabla\alpha + \phi^2\,\nabla\beta + 2n(\alpha^2 + \beta^2)\xi$$

Therefore by (3-3), ξ is harmonic if and only if:

$$\phi\,\nabla\alpha + \phi^2\,\nabla\beta = 0,$$

which is equivalent to:

$$
\begin{aligned}
0 &= \phi^2\,\nabla\alpha - \phi\,\nabla\beta = -\nabla\alpha + \langle\nabla\alpha, \xi\rangle\xi - \phi\,\nabla\beta \\
&= -\nabla\alpha - 2\alpha\beta\xi - \phi\,\nabla\beta, \quad \text{by Theorem 4.7.} \tag{4-10}
\end{aligned}
$$

An α-Sasakian (resp. β-Kenmotsu) structure satisfies (4-10) if and only if α is locally constant (resp. $\nabla\beta$ is characteristic). If $\dim M \geqslant 5$ then (4-10) holds precisely when $\nabla\beta$ is characteristic, by Theorem 4.7.

From Theorem 3.4 and (4-3):

$$
\begin{aligned}
\langle\psi, \Psi\rangle(X) &= \langle R(E_i, X)\xi, \nabla_{E_i}\xi\rangle = -\alpha\langle R(F_i, X)\xi, \phi F_i\rangle + \beta\langle R(F_i, X)\xi, F_i\rangle \\
&= \alpha\mathcal{R}(\Phi)(\xi, X) + \beta\,\mathrm{Ric}(\xi, X) \\
&= \alpha\langle\phi^2\,\nabla\alpha - \phi\,\nabla\beta, X\rangle + \beta\langle\phi\,\nabla\alpha - \phi^2\,\nabla\beta - 2n\,\nabla\beta + 2n(\alpha^2 - \beta^2)\xi, X\rangle,
\end{aligned}
$$

by (4-6) and Lemma 4.6 (a). Inserting the harmonic section equations (4-10), and noting that when $n > 1$ these imply that $\nabla\beta$ is characteristic:

$$
\begin{aligned}
\langle\psi, \Psi\rangle^\sharp &= 2\beta\big(n(\alpha^2 - \beta^2)\xi - \phi^2\,\nabla\beta - n\,\nabla\beta\big) \\
&= 2\beta\big(n(\alpha^2 - \beta^2)\xi + (n-1)\phi^2\,\nabla\beta - n\langle\nabla\beta, \xi\rangle\xi\big) \\
&= 2n\beta\big((\alpha^2 - \beta^2) - (\xi.\beta)\big)\xi, \quad \text{for all } n.
\end{aligned}
$$

Therefore if σ is a harmonic section then:

$$\langle \psi, \Psi \rangle = n\big(2\beta(\alpha^2 - \beta^2) - \xi.(\beta^2)\big)\eta$$

Hence if $\beta = 0$ then every harmonic section is a harmonic map. If $\alpha = 0$ then a harmonic section is a harmonic map if and only if:

$$\beta(\beta^2 + (\xi.\beta)) = 0$$

Thus either $\beta = 0$ or $\xi.\beta = -\beta^2$ (pointwise). Therefore σ is a harmonic map if and only if $\nabla\beta = -\beta^2\xi$. If M is compact then β achieves a maximum (resp. minimum) at x_1 (resp. x_2), say. Therefore $\nabla\beta(x_1) = 0 = \nabla\beta(x_2)$, hence $\beta(x_1) = 0 = \beta(x_2)$, so $\beta = 0$. The characterization of harmonic maps when $\dim M \geqslant 5$ follows from Theorem 4.7. □

Remarks.

(1)   For a Sasakian structure, σ is a harmonic map. This generalizes Example 4.5, and strengthens the result of [25] that the characteristic field on a Sasakian manifold is harmonic. For a Kenmotsu manifold, σ is a harmonic section, but never a harmonic map. For a compact β-Kenmotsu manifold, σ is never a harmonic map.

(2)   Another way of phrasing Theorem 4.8 is that harmonic sections are characterized by $\mathrm{Ric}^*(\xi) = 0$, and harmonic maps are characterized by $\mathrm{Ric}^*(\xi) = 0 = \mathrm{Ric}(\xi)$, where the *∗Ricci curvature* of an almost contact manifold is defined:

$$\mathrm{Ric}^*(X, Y) = \mathcal{R}(\Phi)(X, \phi Y).$$

5. HYPERSURFACES

We continue the discussion from §4 of an oriented hypersurface M of an almost Hermitian manifold $(\tilde{M}, g, J)$, with the induced almost contact structure (4-1). Let $A\colon TM \to TM$ (resp. α) be the shape operator (resp. second fundamental form):

$$AX = -\tilde{\nabla}_X \nu, \qquad \alpha(X, Y) = \langle X, AY \rangle \tag{5-1}$$

5.1 Lemma. *For all $X \in TM$ we have:*

(a) $\tilde{\nabla}_X J(\xi) = \phi^2 AX - J\nabla_X\xi$;   (b) $\tilde{\nabla}_X J(\nu) = \phi AX - \nabla_X\xi$

Proof. First note that $\tilde{\nabla}_X J(\nu)$ is orthogonal to the holomorphic 2-plane $\xi \wedge \nu$. For any Z in $\mathcal{D}$ it follows from (5-1) and (4-1) that:

$$\langle \tilde{\nabla}_X J(\nu), Z \rangle = \langle -\tilde{\nabla}_X\xi - J\tilde{\nabla}_X\nu, Z \rangle = \langle \phi AX - \nabla_X\xi, Z \rangle$$

This establishes (b), from which (a) follows since $\xi = -J\nu$.   □

We introduce the following skew-symmetric $(1,1)$ tensor Γ_1 on M:

$$\Gamma_1(X) = \tilde{\nabla}_\nu J(X) - \langle \tilde{\nabla}_\nu J(X), \nu \rangle \nu, \quad \text{for all } X \in TM. \tag{5-2}$$

5.2 Proposition. *For all X in TM and Z in $\mathcal{D}$ the components of $\tilde{\nabla}^2_{X,X} J(Z)$ are:*

(a) In $\mathcal{D}$:

$$\bar{\nabla}^2_{X,X} J(Z) + \alpha(X,X)\,\phi^2\Gamma_1 Z + r(\nabla_X\xi, J\,\nabla_X\xi)Z$$
$$- \, r(\phi^2 AX, \phi AX)Z + 2r(\phi^2 AX, \nabla_Y\xi)Z$$

(b) *In the ξ-direction:*

$$\nabla_X\alpha(X,Z) - \langle\nabla^2_{X,X}\xi, JZ\rangle + \alpha(X,X)\langle\Gamma_1\xi, Z\rangle + 2\langle(\bar{\nabla}J\otimes\nabla\xi)(X,X), Z\rangle$$
$$- \, \alpha(X,\xi)\alpha(X,JZ) - 2\alpha(X,\xi)\langle\nabla_X\xi, Z\rangle$$

(c) *In the ν-direction:*

$$\nabla_X\alpha(X,JZ) + \langle\nabla^2_{X,X}\xi, Z\rangle + \alpha(X,X)\langle\tilde{\nabla}_\nu J(\nu), Z\rangle + 2\alpha(\bar{\nabla}_X J(Z), X)$$
$$+ \, \alpha(X,\xi)\alpha(Y,Z) - 2\alpha(X,\xi)\langle\nabla_X\xi, JZ\rangle$$

Proof. If Z is extended to a vector field then:

$$\tilde{\nabla}^2_{X,X} J(Z) = [\tilde{\nabla}^2_{X,X}, J]Z - 2\,\tilde{\nabla}_X J(\tilde{\nabla}_X Z) \tag{5-3}$$

For convenience, extend $X \in T_xM$ to a vector field on M with $\nabla X(x) = 0$, and Z to a section of $\mathcal{D}$ with $\bar{\nabla}Z(x) = 0$. Then:

$$\tilde{\nabla}_X X = \alpha(X,X)\nu, \qquad \tilde{\nabla}_X Z = \alpha(X,Z)\nu - \langle\nabla_X\xi, Z\rangle\xi$$

It follows that:

$$[\tilde{\nabla}^2_{X,X}, J]Z = \tilde{\nabla}_X\tilde{\nabla}_X(JZ) - \tilde{\nabla}_{\tilde{\nabla}_X X}(JZ) - J(\tilde{\nabla}_X\tilde{\nabla}_X Z - \tilde{\nabla}_{\tilde{\nabla}_X X}Z)$$
$$= [\tilde{\nabla}_X\tilde{\nabla}_X, J]Z - \alpha(X,X)\,\tilde{\nabla}_\nu J(Z)$$

Routine calculations utilizing (4-1) and (5-1) then yield:

$$\tilde{\nabla}_X\tilde{\nabla}_X(JZ) = \tilde{\nabla}_X\big(\bar{\nabla}_X(JZ) - \langle JZ, \nabla_X\xi\rangle\xi + \alpha(X,JZ)\nu\big)$$
$$= \bar{\nabla}^2_{X,X}(JZ) - \langle\nabla_X\xi, JZ\rangle\nabla_X\xi - \alpha(X,JZ)AX$$
$$- \, \langle\nabla^2_{X,X}\xi, JZ\rangle\xi + 2\langle\bar{\nabla}_X J(\nabla_X\xi), Z\rangle\xi$$
$$+ \, \nabla_X\alpha(X,JZ)\nu + 2\alpha(X, \bar{\nabla}_X J(Z))\nu - 2\alpha(X,\xi)\langle\nabla_X\xi, JZ\rangle\nu,$$

$$J\tilde{\nabla}_X\tilde{\nabla}_X Z = J\tilde{\nabla}_X\big(\bar{\nabla}_X Z - \langle Z, \nabla_X\xi\rangle\xi + \alpha(X,Z)\nu\big)$$
$$= J\bar{\nabla}^2_{X,X} Z - \langle\nabla_X\xi, Z\rangle J\,\nabla_X\xi - \alpha(X,Z)\phi AX$$
$$- \, \nabla_X\alpha(X,Z)\xi + 2\alpha(X,\xi)\langle\nabla_X\xi, Z\rangle\xi - \langle\nabla^2_{X,X}\xi, Z\rangle\nu - \alpha(X,\xi)\alpha(X,Z)\nu$$

The terms in (5-3) involving $\tilde\nabla J$ may be evaluated using Lemma 5.1, and the components of $\tilde\nabla^2 J$ extracted. For the $\mathcal{D}$-component, it follows from (5-3) with $\bar\nabla$ in place of $\tilde\nabla$ that:

$$\bar\nabla^2_{X,X}(JZ) - J\,\bar\nabla^2_{X,X}Z = \bar\nabla^2_{X,X}J(Z)$$

The $\mathcal{D}$-component then simplifies using (3-5). $\qquad\qquad\square$

We denote by H the (scalar) mean curvature of M:

$$(2n+1)H = \alpha(E_i, E_i),$$

and introduce a second skew-symmetric $(1,1)$ tensor Γ_2 on M:

$$\Gamma_2(X) = \tilde\nabla^2_{\nu,\nu}J(X) - \langle\tilde\nabla^2_{\nu,\nu}J(X),\nu\rangle\nu, \quad \text{for all } X \in TM. \tag{5-4}$$

We also abbreviate:

$$\langle\bar\nabla J, A\rangle(Z) = \langle\bar\nabla_{E_i}J(Z), AE_i\rangle, \qquad \tilde\tau(J) = \tfrac{1}{4}\left[\tilde\nabla^*\tilde\nabla J, J\right]$$

5.3 Proposition. *For all Z, W in $\mathcal{D}$ the non-zero components of $\tilde\tau(J)|_M$ are given by:*

(a) $\quad 2\langle\tilde\tau(J)Z, W\rangle = 2\langle\bar\tau(J)Z, W\rangle - \tfrac{1}{2}\langle[\Gamma_2, \phi]Z, W\rangle + (2n+1)H\langle\Gamma_1 Z, JW\rangle$
$$+ \langle J\,\nabla_{AZ}\xi - \nabla_{AJZ}\xi, W\rangle - \langle J\,\nabla_{AW}\xi - \nabla_{AJW}\xi, Z\rangle$$

(b) $\quad 2\langle\tilde\tau(J)Z, \xi\rangle = \nabla^*\alpha(JZ) - \langle\bar\nabla J, A\rangle(Z) + \langle\nabla^*\nabla\xi + JT(\phi), Z\rangle$
$$- \langle A\xi, AZ\rangle + 2\langle\nabla_{A\xi}\xi, JZ\rangle$$

Proof. Since $r(E, JE)$ commutes with J, Proposition 5.2 (a) implies:

$$4\langle\tilde\tau(J)Z, W\rangle = 4\langle\bar\tau(J)Z, W\rangle - \langle[\tilde\nabla^2_{\nu,\nu}J, J]Z, W\rangle - (2n+1)H\langle[\phi^2\Gamma_1, J]Z, W\rangle$$
$$+ 2\sum_i\langle[r(AE_i, \nabla_{E_i}\xi), J]Z, W\rangle,$$

which expands as stated, noting that:

$$\langle[\phi^2\Gamma_1, J]Z, W\rangle = -\langle[\Gamma_1, \phi]Z, W\rangle = -2\langle\Gamma_1 Z, JW\rangle,$$
$$\langle[\tilde\nabla^2_{\nu,\nu}J, J]Z, W\rangle = \langle\Gamma_2(JZ), W\rangle + \langle\Gamma_2 Z, JW\rangle = \langle[\Gamma_2, \phi]Z, W\rangle$$

On the other hand since $J\xi = \nu$ we have:

$$4\langle\tilde\tau(J)Z, \xi\rangle = \langle\tilde\nabla^*\tilde\nabla J(JZ), \xi\rangle + \langle\tilde\nabla^*\tilde\nabla J(Z), \nu\rangle,$$

and by Proposition 5.2 (b) and (c):

$$\langle \tilde{\nabla}^* \tilde{\nabla} J(JZ), \xi \rangle + 2\langle T(\phi), JZ \rangle = \nabla^* \alpha(JZ) + \langle \nabla^* \nabla \xi, Z \rangle - \langle A\xi, AZ \rangle + 2\langle \nabla_{A\xi} \xi, JZ \rangle$$
$$= \langle \tilde{\nabla}^* \tilde{\nabla} J(Z), \nu \rangle + 2\langle \bar{\nabla} J, A \rangle(Z)$$

Finally note that (a) and (b) describe all the non-zero components of $\tilde{\tau}(J)$, because:

$$\langle \tilde{\tau}(J)Z, \nu \rangle = \langle \tilde{\tau}(J)(JZ), \xi \rangle \quad \text{and} \quad \langle \tilde{\tau}(J)\xi, \nu \rangle = 0. \qquad \square$$

Our first application of Proposition 5.3 is when $\tilde{M} = M \times \mathbb{R}$. Then $\tilde{M}$ acquires a canonical almost Hermitian structure J, by applying (4-1) to each leaf of the canonical hypersurface foliation, with $\nu = d/dt$. Let $\tilde{\sigma}$ be the section of the twistor bundle over $\tilde{M}$ parametrizing J.

5.4 Theorem. *The almost Hermitian structure on $M \times \mathbb{R}$ is harmonic if and only if:*

$$\bar{\tau}(J) = 0 \qquad \text{and} \qquad \nabla^* \nabla \xi = |\nabla \xi|^2 \xi - JT(\phi)$$

Furthermore:

$$\langle \tilde{\psi}, \tilde{\Psi} \rangle = \langle \psi_1, \Psi_1 \rangle + \tfrac{1}{2} \langle \psi_2, \Psi_2 \rangle$$

If any two of ξ, σ, $\tilde{\sigma}$ are harmonic sections (resp. maps) then so is the third.

Proof. We have $\alpha = 0$, and $\Gamma_1 = 0 = \Gamma_2$, by the geometry of the product metric. Proposition 5.3 therefore reduces to:

$$\langle \tilde{\tau}(J)Z, W \rangle = \langle \bar{\tau}(J)Z, W \rangle, \qquad 2\langle \tilde{\tau}(J)Z, \xi \rangle = \langle \nabla^* \nabla \xi + JT(\phi), Z \rangle,$$

and since the ξ-component of $\nabla^* \nabla \xi$ is $|\nabla \xi|^2$ it follows that $\tilde{\tau}(J) = 0$ if and only if:

$$\bar{\tau}(J) = 0 \qquad \text{and} \qquad \nabla^* \nabla \xi - |\nabla \xi|^2 \xi + JT(\phi) = 0$$

Let $\{F_i\}$ be a local orthonormal frame of $\mathcal{D}$, extended to a local frame $\{E_i\}$ of TM by defining $E_{2n+1} = \xi$, and a local frame $\{\tilde{E}_i\}$ of $T\tilde{M}$ by defining $\tilde{E}_{2n+2} = \nu$. Since $\tilde{R}$ vanishes when applied to ν (in any of its arguments), for all $X \in TM$ we have:

$$4\langle \tilde{\psi}, \tilde{\Psi} \rangle(X) = \langle \tilde{R}(\tilde{E}_i, X)\tilde{E}_j, J\tilde{\nabla}_{\tilde{E}_i} J(\tilde{E}_j) \rangle = \langle R(E_i, X)E_j, J\tilde{\nabla}_{E_i} J(E_j) \rangle$$
$$= \langle R(E_i, X)\xi, \nabla_{E_i} \xi \rangle + \langle R(E_i, X)F_j, J\tilde{\nabla}_{E_i} J(F_j) \rangle, \quad \text{by Lemma 5.2,}$$
$$= \langle \psi_2, \Psi_2 \rangle(X) + 4\langle \psi_1, \Psi_1 \rangle(X)$$
$$\qquad - \langle R(E_i, X)F_j, \langle \tilde{\nabla}_{E_i} J(F_j), \nu \rangle \xi \rangle, \quad \text{by Theorem 3.4 and Lemma 4.2,}$$
$$= 4\langle \psi_1, \Psi_1 \rangle(X) + 2\langle \psi_2, \Psi_2 \rangle(X), \quad \text{by Lemma 5.2.}$$

The relationships between the harmonicity of ξ, σ and $\tilde{\sigma}$ follow by comparison with Theorems 3.2 and 3.4. $\qquad \square$

The following consequence of Theorems 5.4 and 4.8 generalizes the result of [27] that the almost Hermitian structure on the Hopf manifold $S^{2n+1} \times S^1$ is harmonic.

5.5 Corollary. *Suppose M is a Sasakian manifold. Then the section $\tilde{\sigma}$ parametrizing the canonical almost Hermitian structure on $M \times \mathbb{R}$ (or $M \times S^1$) is a harmonic map.*

Recall that an almost contact manifold is said to be *nearly cosymplectic* [2] if:

$$\nabla_X \phi(X) = 0, \qquad \text{for all } X \in TM. \tag{5-5}$$

5.6 Corollary. *Suppose M is a nearly cosymplectic manifold with parallel characteristic field. Then σ is a harmonic map.*

Proof. It follows from (5-5) that $\tilde{\nabla}_X J(X) = 0$ for all $X \in TM$. Moreover:

$$\tilde{\nabla}_\nu J(X) = \Gamma_1(X) = 0, \qquad \tilde{\nabla}_X J(\nu) = -\nabla_X \xi,$$

so $\tilde{M}$ is nearly Kähler precisely when ξ is parallel. Now $\tilde{\sigma}$ is a harmonic map [29], and therefore σ is a harmonic map by Theorem 5.4.   □

By analogy with [14] we say a nearly cosymplectic structure is *strict* if $\nabla_X \phi \neq 0$ for all non-zero X; for example, if ϕ is induced by a strict nearly Kähler structure on $\tilde{M}$. Corollary 5.6 does not apply if ϕ is strict, because nearly cosymplectic manifolds satisfy:

$$\nabla_\xi \phi(X) = -\phi \nabla_X \xi$$

We therefore consider the case when $\tilde{M}$ is nearly Kähler. We may (locally) extend ν to a vector field on a neighbourhood of M in $\tilde{M}$ by parallel translation along the normal geodesics. Any vector field on M may be extended in a similar way. Since $\tilde{M}$ is nearly Kähler its geodesics are holomorphically planar. In particular, the holomorphic 2-plane $\xi \wedge \nu$ remains holomorphic when parallel transported along a ν-geodesic, and hence:

$$\tilde{\nabla}_\nu J(\nu) = -\tilde{\nabla}_\nu \xi - J \tilde{\nabla}_\nu \nu = 0 \tag{5-6}$$

Consequently, by Lemma 5.1 (b), the definition (5-2) of Γ_1 simplifies to:

$$\Gamma_1 X = \tilde{\nabla}_\nu J(X) = \nabla_X \xi - \phi AX \tag{5-7}$$

Furthermore Γ_1 is a section of $\mathfrak{M}_1$ (see §3), for it follows from (4-1) that:

$$\{\Gamma_1, \phi\}X = -\langle X, \tilde{\nabla}_\nu J(\nu)\rangle \xi - \eta(X)\,\tilde{\nabla}_\nu J(\nu) = 0.$$

5.7 Proposition. *Suppose $\tilde{M}$ is nearly Kähler.*

(a)   *The integral curves of ξ are geodesics if and only if ξ is a principal direction.*

(b)   *The characteristic field ξ is Killing if and only if $[A, \phi] = 0$.*

(c)   *M inherits a contact metric structure if and only if $\Gamma_1 = 0$ and $\{A, \phi\} = -2\phi$.*

(d)   *M inherits a nearly cosymplectic structure if and only if $\alpha = (2n+1)H\eta \otimes \eta$.*

Proof.

(a)   Since $\Gamma_1 \xi = 0$ it follows from (5-7) that $\nabla_\xi \xi = \phi A \xi$.

(b)   It follows from (5-7) and the skew-symmetry of Γ_1 that:

$$\langle \nabla_X \xi, Y \rangle + \langle X, \nabla_Y \xi \rangle = \langle [\phi, A] X, Y \rangle$$

(c)   From (5-7):

$$\begin{aligned} 2d\eta(X, Y) &= \nabla_X \eta(Y) - \nabla_Y \eta(X) = \langle \nabla_X \xi, Y \rangle - \langle X, \nabla_Y \xi \rangle \\ &= -\langle X, (2\Gamma_1 + \{A, \phi\}) Y \rangle \end{aligned} \tag{5-8}$$

Therefore if $\Gamma_1 = 0$ and $\{A, \phi\} = -2\phi$ then ϕ is a contact metric structure. Conversely, since contact metric structures satisfy $\nabla_\xi \phi = 0$ it follows from (5-7), Lemma 4.2 and (4-2) that for all $Z \in \mathcal{D}$:

$$\Gamma_1 Z = \tilde{\nabla}_\nu J(Z) = -\tilde{\nabla}_\xi J(JZ) = -\bar{\nabla}_\xi J(JZ) = -\nabla_\xi \phi(JZ) = 0 \tag{5-9}$$

Hence $\Gamma_1 = 0$. It then follows from (5-8) that $\{A, \phi\} = -2\phi$.

(d)   Proved in [3, Theorem 6.13].   □

Proposition 5.7 (c) generalizes the criterion [20] for contact metric hypersurfaces of Kähler manifolds. In particular, if $\tilde{M}$ is a strict nearly Kähler manifold then no contact metric hypersurfaces exist.

Concerning Γ_2, it follows from (5-6) that:

$$\tilde{\nabla}^2_{\nu,\nu} J(\nu) = \tilde{\nabla}_\nu \tilde{\nabla}_\nu J(\nu) = \tilde{\nabla}_\nu (\tilde{\nabla}_\nu J(\nu)) - \tilde{\nabla}_\nu J(\tilde{\nabla}_\nu \nu) = 0,$$

so (5-4) simplifies to:

$$\Gamma_2(X) = \tilde{\nabla}^2_{\nu,\nu} J(X), \quad \text{for all } X \in TM.$$

Since $\tilde{M}$ is nearly Kähler, it follows from [13, Proposition 2.3] that:

$$\langle \tilde{\nabla}^2_{\nu,\nu} J(-), J(-) \rangle = -\langle \tilde{\nabla}_\nu J(-), \tilde{\nabla}_\nu J(-) \rangle,$$

and hence, for all $X, Y \in TM$:

$$\langle \Gamma_2 X, Y \rangle = \langle \phi \Gamma_1 X, \Gamma_1 Y \rangle$$

Therefore, since Γ_1 is a section of $\mathfrak{M}_1$, Γ_2 is a section of $\mathfrak{H}$.

5.8 Proposition. *If $\tilde{M}$ is nearly Kähler then for all $Z, W \in \mathcal{D}$ we have:*

(a) $\langle \bar{\tau}(J)Z, W \rangle = 2(2n+1)H \langle J\Gamma_1 Z, W \rangle - 2 \langle [\{A, \phi\}, \Gamma_1]Z, W \rangle$

(b) $\langle \nabla^*\nabla\xi + JT(\phi), Z \rangle = \langle (\Gamma_1 J - A)Z, A\xi \rangle - \nabla^*\alpha(JZ)$

Proof. Since $\tilde{M}$ is nearly Kähler we have $\tilde{\tau}(J) = 0$ [27].

(a)   Since Γ_2 commutes with ϕ, Proposition 5.3 (a) reduces to:

$$-\tfrac{1}{2}\langle \bar{\tau}(J)Z, W \rangle = (2n+1)H \langle \Gamma_1 Z, JW \rangle + \Sigma,$$

where:

$$\Sigma = \langle \nabla_{AW}\xi, JZ \rangle - \langle \nabla_{AZ}\xi, JW \rangle + \langle \nabla_{AJW}\xi, Z \rangle - \langle \nabla_{AJZ}\xi, W \rangle$$

By (5-7), noting the cancellation of terms involving A^2, we obtain:

$$\begin{aligned}
\Sigma &= \langle \Gamma_1 AW, JZ \rangle - \langle \Gamma_1 AZ, JW \rangle + \langle \Gamma_1 AJW, Z \rangle - \langle \Gamma_1 AJZ, W \rangle \\
&= \langle (\phi\Gamma_1 A - \Gamma_1 A\phi + \phi A\Gamma_1 - A\Gamma_1\phi)Z, W \rangle, \quad \text{by (4-1)} \\
&= \langle [\{A, \phi\}, \Gamma_1]Z, W \rangle, \quad \text{since } \Gamma_1 \text{ anticommutes with } \phi.
\end{aligned}$$

(b)   Define a symmetric tensor $S \colon \mathcal{D} \to \mathcal{D}$ by $S = \phi^2 A | \mathcal{D}$. Since $\tilde{M}$ is nearly Kähler:

$$\langle \bar{\nabla}J(Z), A \rangle = \langle \bar{\nabla}_Z J, S \rangle + \langle \bar{\nabla}_\xi J(Z), A\xi \rangle = \langle J\Gamma_1 Z, A\xi \rangle,$$

by the skew-symmetry of $\bar{\nabla}_Z J \colon \mathcal{D} \to \mathcal{D}$, and (5-9). Furthermore, by (5-7):

$$\langle \nabla_{A\xi}\xi, JZ \rangle = \langle (A - \Gamma_1 J)Z, A\xi \rangle,$$

and Proposition 5.3 (b) simplifies as stated.   □

In view of Proposition 5.7 it is reasonable to consider hypersurfaces with $\Gamma_1 = 0$ and ξ a principal direction, which include all contact metric hypersurfaces; in fact, scrutiny of the proof shows they are characterized by the intrinsic condition $\nabla_\xi \phi = 0$. We say that ϕ is *harmonic* if σ is a harmonic section.

5.9 Theorem. *Let M be a hypersurface of a nearly Kähler manifold $\tilde{M}$, with $\Gamma_1 = 0$ and ξ a principal direction. The induced almost contact structure ϕ is harmonic if and only if ξ is harmonic, if and only if $\nabla^*\alpha$ is characteristic. If ν (equivalently, ξ) is a Ricci-principal direction in $\tilde{M}$ then ϕ is harmonic precisely when ∇H is characteristic. Thus if $\tilde{M}$ is Einstein and M has constant mean curvature then ϕ is harmonic.*

Proof. Since ξ is a principal direction, we may pick an orthonormal frame $\{F_i\}$ of $\mathcal{D}$ with $AF_i = \kappa_i F_i$, say. Then since $\bar{\nabla}_\xi J = 0$ by (5-9), it follows from (5-7) that:

$$T(\phi) = \bar{\nabla}_{F_i}J(\nabla_{F_i}\xi) = \bar{\nabla}_{F_i}J(\phi AF_i) = -\kappa_i J\bar{\nabla}_{F_i}J(F_i) = 0,$$

by Lemma 4.2. Furthermore $\bar{\tau}(J) = 0$ by Proposition 5.8 (a). It therefore follows from Theorem 3.2 that ϕ is harmonic if and only if ξ is harmonic. But by Proposition 5.8 (b) this is the case precisely when $\nabla^*\alpha(\mathcal{D}) = 0$.

It follows from Codazzi's equation:

$$\nabla_X \alpha(Y, Z) - \nabla_Y \alpha(X, Z) = \langle \tilde{R}(X, Y)Z, \nu \rangle,$$

that:

$$\nabla^*\alpha(Z) = \widetilde{\mathrm{Ric}}(Z, \nu) - (2n + 1)dH(Z) = -\widetilde{\mathrm{Ric}}(Z, \xi) - (2n + 1)dH(Z),$$

since the Ricci curvature of a nearly Kähler manifold is J-invariant [18]. The J-invariance also implies $\widetilde{\mathrm{Ric}}(\nu, \xi) = 0$, and therefore ν (equivalently, ξ) is an eigenvector of $\widetilde{\mathrm{Ric}}$ if and only if $\widetilde{\mathrm{Ric}}(\nu, \mathcal{D}) = 0$ (equivalently, $\widetilde{\mathrm{Ric}}(\xi, \mathcal{D}) = 0$). Therefore $\nabla^*\alpha$ is characteristic if and only if dH is characteristic. $\qquad\square$

5.10 Theorem. *Suppose M is a totally umbilical hypersurface of a nearly Kähler manifold $\tilde{M}$. If M is totally geodesic and ξ is harmonic then the induced almost contact metric structure is harmonic. The converse is true if $\tilde{M}$ is strict.*

Proof. Since $\alpha = Hg$, Proposition 5.8 reduces to:

$$\bar{\tau}(J) = 2(2n - 3)HJ\Gamma_1 \tag{5-10}$$

$$\nabla^*\nabla\xi - |\nabla\xi|^2\xi + JT(\phi) = -\phi\nabla H \tag{5-11}$$

Therefore if $H = 0$ and ξ is harmonic then $\bar{\tau}(J) = 0$ and $T(\phi) = 0$, so ϕ is harmonic by Theorem 3.2. Conversely if $\tilde{M}$ is strict and ϕ is harmonic then (5-10) implies $H = 0$, and comparison of (5-11) with Theorem 3.2 implies ξ is harmonic. $\qquad\square$

It follows from Theorem 5.10 that the only (round) hypersphere of the nearly Kähler 6-sphere whose induced almost contact structure is harmonic is the equator, which is nearly cosymplectic. In particular, the almost contact structure induced on the hypersphere of radius $1/\sqrt{2}$, which is *nearly Sasakian* [5], is not harmonic. Interestingly, the characteristic vector field of any round hypersphere is Killing (eg. Proposition 5.7), and hence harmonic (cf. Theorem 4.3). Regarding the nearly cosymplectic 5-sphere:

Theorem 5.11. *The section parametrizing the standard nearly cosymplectic structure on S^5 is a harmonic map.*

Proof. Let $\{F_i\}$ be an orthornormal frame in $\mathcal{D}$, extended to a frame $\{E_i\}$ of TM with $E_{2n+1} = \xi$. Note that (5-5) implies $\bar{\nabla}J(F, F) = 0$ for all $F \in \mathcal{D}$. From Theorem 3.4:

$$4\langle \psi_1, \Psi_1 \rangle(X) = \langle R(E_i, X)F_j, J\bar{\nabla}_{E_i}J(F_j) \rangle = \langle \langle X, F_j \rangle E_i - \langle E_i, F_j \rangle X, J\bar{\nabla}_{E_i}J(F_j) \rangle$$
$$= -2\langle J\bar{\nabla}_{F_i}J(F_i), X \rangle = 0,$$
$$\langle \psi_2, \Psi_2 \rangle(X) = \langle R(E_i, X)\xi, \nabla_{E_i}\xi \rangle = \langle X, \xi \rangle \operatorname{div}\xi - \langle X, \nabla_\xi \xi \rangle = 0. \qquad \square$$

References

[1] D. E. Blair, *The theory of quasi-Sasakian structures*, J. Diff. Geom. **1** (1967), 331–345.

[2] D. E. Blair, *Almost contact manifolds with Killing structure tensors*, Pacific J. Math. **39** (1971), 373–379.

[3] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, vol. 203, Birkhauser, 2002.

[4] D. E. Blair & J. A. Oubina, *Conformal and related changes of metric on the product of two almost contact mketric manifolds*, Publ. Mat. **34** (1990), 199–207.

[5] D. E. Blair, D. K. Showers & K. Yano, *Nearly Sasakian structures*, Kōdai Math. Sem. Rep. **27** (1976), 175–180.

[6] P. M. Chacón, A. M. Naveira & J. M. Weston, *On the energy of distributions, with application to the quaternionic Hopf fibration*, Monatshefte Math. **133** (2001), 281–294.

[7] B-Y. Choi & J-W. Yim, *Distributions on Riemannian manifolds, which are harmonic maps*, Tôhoku Math. J. **55** (2003), 175–188.

[8] J. Davidov & O. Muskarov, *Harmonic almost-complex structures on twistor space*, Israel J. Math. **131** (2002), 319–332.

[9] U. C. De & M. M. Tripathi, *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook Math. J. **43** (2003), 1–9.

[10] O. Gil-Medrano, *Relationship between volume and energy of unit vector fields*, Diff. Geom. Appl. **15** (2001), 137–152.

[11] O. Gil-Medrano, *Unit vector fields that are critical points of the volume and of the energy: characterization and examples*, Complex, Contact and Symmetric Manifolds (O. Kowalski, E. Musso, D. Perrone, eds.), Progress in Mathematics, vol. 234, Birkhauser, 2005, pp. 165–186.

[12] J. C. Gonzalez-Davila & L. Vanhecke, *Energy and volume of unit vector fields on three-dimensional Riemannian manifolds*, Diff. Geom. Appl. **16** (2002), 225–244.

[13] A. Gray, *Riemannian manifolds with geodesic symmetries of order 3*, J. Diff. Geom. **7** (1972), 343–369.

[14] A. Gray, *The structure of nearly Kähler manifolds*, Math. Ann. **223** (1976), 233–248.

[15] J. Gray, *Some global properties of contact structures*, Annals of Math. **69** (1959), 421–450.

[16] S-D. Han & J-W. Yim, *Unit vector fields on spheres which are harmonic maps*, Math. Z. **27** (1998), 83–92.

[17] J-S. Kim, R. Prasad & M. M. Tripathi, *On generalized Ricci-recurrent trans-Sasakian manifolds*, J. Korean Math. Soc. **39** (2002), 953–961.

[18]   S. Kotō, *Some theorems on almost Kaehlerian spaces*, J. Math. Soc. Japan **12** (1960), 422–433.

[19]   J. C. Marrero, *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. **162** (1992), 77–86.

[20]   M. Okumura, *Contact hypersurfaces in certain Kaehlerian manifolds*, Tôhoku Math. J. **18** (1966), 74–102.

[21]   J. A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), 187–193.

[22]   D. Perrone, *Contact metric manifolds whose characteristic vector field is a harmonic vector field*, Diff. Geom. Appl. **20** (2004), 367–378.

[23]   P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

[24]   E. Vergara Diaz, *Harmonic Sections and Almost Contact Geometry*, PhD Thesis, University of York, 2005.

[25]   G. Wiegmink, *Total bending of vector fields on Riemannian manifolds*, Math Ann. **303** (1995), 325–344.

[26]   C. M. Wood, *A class of harmonic almost product structures*, J. Geom. Phys. **14** (1994), 25–42.

[27]   C. M. Wood, *Harmonic almost complex structures*, Compositio Math. **99** (1995), 183–212.

[28]   C. M. Wood, *On the energy of a unit vector field*, Geom. Dedicata **64** (1997), 319–330.

[29]   C. M. Wood, *Harmonic sections of homogeneous fibre bundles*, Diff. Geom. Appl. **19** (2003), 193–210.

Department of Mathematics, University of York, Heslington, York Y010 5DD, UK.

E-mail address: evd103@york.ac.uk, cmw4@york.ac.uk